

# Hitchin's Connection in Half-form Quantization

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## Abstract

We give a differential geometric construction of a connection in the bundle of quantum Hilbert spaces arising from half-form corrected geometric quantization of a prequantizable, symplectic manifold, endowed with a rigid, family of Kähler structures, all of which give vanishing first Dolbeault cohomology groups.

In [And1] Andersen gave an explicit construction of Hitchin's connection in the non-corrected case using additional assumptions. Under the same assumptions we also give an explicit solution in terms of Ricci potentials. Moreover we show that if these are carefully chosen the construction coincides with the construction of Andersen in the non-corrected case.

## 1 Introduction

Hitchin constructed in [Hit] a connection over Teichmüller space. This Hitchin connection is a connection in the bundle obtained from geometric quantization of the moduli spaces of flat  $SU(n)$ -connections on a closed oriented surface. The significance of this connection is its relation to  $(2+1)$ -dimensional Reshetikhin-Turaev TQFT ([RT1] and [RT2]). In fact, this geometric construction of these TQFT's was proposed by Witten in [Wit], where he derived, via the Hamiltonian approach to quantum Chern-Simons theory, that the geometric quantization of the moduli spaces of flat connections should give the two dimensional part of the theory. Further, he proposed an alternative construction of the two dimensional part of the theory via WZW-conformal field theory. This theory has been studied intensively. In particular the work of Tsuchiya, Ueno and Yamada in [TUY] provided the major geometric constructions and results needed. In [BK], their results was used to show that

the category of integrable highest weight modules of level  $k$  for the affine Lie algebra associated to any simple Lie algebra is a modular tensor category. Further in [BK] this result is combined with the work of Kazhdan and Lusztig ([KL1], [KL2] and [KL3]) and the work of Finkelberg [Fin] to argue that this category is isomorphic to the modular tensor category associated to the corresponding quantum group, from which Reshetikhin and Turaev constructed their TQFT. Unfortunately, these results do not allow one to conclude the validity of the geometric constructions of the two dimensional part of the TQFT proposed by Witten. However, in joint work with Ueno, [AU1], [AU2], [AU3] and [AU4], we have given a proof, based mainly on the results of [TUY], that the TUY-construction of the WZW-conformal field theory after twist by a fractional power of an abelian theory, satisfies all the axioms of a modular functor. Furthermore, we have proved that the full  $2 + 1$ -dimensional TQFT that results from this is isomorphic to the one constructed by BHMV via skein theory mentioned above. Combining this with the Theorem of Laszlo [Las], which identifies (projectively) the representations of the mapping class groups one obtains from the geometric quantization of the moduli space of flat connections with the ones obtained from the TUY-constructions, one gets a proof of the validity of the construction proposed by Witten in [Wit].

In [ADPW], Axelrod, Della Pietra and Witten gave a differential geometric construction of the Hitchin connection by using a method of symplectic reduction from the infinite dimensional space of all  $SU(n)$ -connections. In [And1] Andersen constructed the Hitchin connection in a more general setting. A corollary of the results in [And1] is that the connection constructed by Axelrod, Della Pietra and Witten in [ADPW] is the same as Hitchin's connection constructed in [Hit].

In this paper, we extend the setting from [And1], in which we can construct the Hitchin connection. Let us describe this setting.

Let  $(M, \omega)$  be a symplectic manifold of dimension  $2m$ , and let  $J$  be a family of Kähler structures on  $M$ , parametrized smoothly by a manifold  $\mathcal{T}$ . Along any vector field  $V$  on  $\mathcal{T}$ , we can differentiate  $J$  to get a map  $V[J]: \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM_{\mathbb{C}}))$ .

Define  $\tilde{G}(V) \in C^\infty(M, S^2(TM_{\mathbb{C}}))$  by

$$V[J] = \tilde{G}(V)\omega.$$

Letting  $T_\sigma$  denote the holomorphic tangent bundle on  $(M, J_\sigma)$  for any  $\sigma \in \mathcal{T}$ , we can further define  $G(V) \in C^\infty(M, S^2(T))$  by the equation

$$\tilde{G}(V) = G(V) + \bar{G}(V),$$

for all real vector field  $V$  on  $\mathcal{T}$ . We shall assume that the family  $J$  is *rigid*, in the sense of Definition 11, meaning that  $G(V)_\sigma$  is a holomorphic section of  $S^2(T_\sigma)$ .

In case the second Stiefel-Whitney class vanishes, we can choose a metaplectic structure on  $(M, \omega)$ , which gives rise to a choice of a square root  $\delta_\sigma$  of the canonical line bundle  $K_\sigma \rightarrow M_\sigma$ , varying smoothly in the parameter  $\sigma \in \mathcal{T}$ .

Now assume that  $(M, \omega)$  is prequantizable in the sense that there exists a Hermitian line bundle  $\mathcal{L}$  with a compatible connection  $\nabla^\mathcal{L}$  of curvature

$$R_{\nabla^\mathcal{L}} = -i\omega.$$

The Levi-Civita connection  $\tilde{\nabla}_\sigma$ , corresponding to the Kähler metric on  $M_\sigma$ , induces a connection in the line bundle  $\delta_\sigma \rightarrow M_\sigma$ , and thus we get a connection  $\nabla_\sigma$  in  $L^k \otimes \delta_\sigma \rightarrow M_\sigma$  giving this bundle the structure of a holomorphic line bundle.

For every  $\sigma \in \mathcal{T}$ , we have the infinite dimensional vector space  $\mathcal{H}_\sigma^{(k)} = C^\infty(M, \mathcal{L}^k \otimes \delta_\sigma)$ , and we consider subspace of holomorphic sections

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) = \{s \in C^\infty(M, \mathcal{L}^k \otimes \delta_\sigma) \mid \nabla_\sigma^{0,1}s = 0\}.$$

It is not clear that the spaces  $\mathcal{H}_\sigma^{(k)}$  form a smooth vector bundle  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ . However, it is a corollary of our construction that, under the assumptions stated in Theorem 1, the spaces  $\mathcal{H}_\sigma^{(k)}$  indeed form a smooth bundle over  $\mathcal{T}$  and that  $H^{(k)} \rightarrow \mathcal{T}$  is a smooth subbundle of  $\mathcal{H}^{(k)}$ .

In the bundle  $C^\infty(M, T) \rightarrow \mathcal{T}$ , we have a connection  $\hat{\nabla}^T$  defined by the formula

$$\hat{\nabla}_V^T \zeta = \pi^{1,0} V[\zeta],$$

where  $\pi_\sigma^{1,0}: TM_\mathbb{C} \rightarrow T_\sigma$  is the projection, and  $V[\zeta]$  denotes differentiation in the trivial bundle  $\mathcal{T} \times C^\infty(M, TM_\mathbb{C})$  (see section 3 for further details). This induces a connection in  $C^\infty(M, \delta) \rightarrow \mathcal{T}$ , and with the help of the trivial connection in  $\mathcal{T} \times C^\infty(M, \mathcal{L}^k)$  this induces a connection  $\hat{\nabla}^r$  in  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ , called the reference connection.

**Theorem 1.** *Let  $(M, \omega)$  be a prequantizable, symplectic manifold with vanishing second Stiefel-Whitney class. Further let  $J$  be a rigid family of Kähler structures on  $M$  all satisfying  $H^{0,1}(M) = 0$ . Then for any vector field  $V$  on  $\mathcal{T}$ , the Hitchin connection  $\nabla$  in the bundle  $H^{(k)}$  is given by*

$$\nabla_V = \hat{\nabla}_V^r + \frac{1}{4k}(\Delta_{G(V)} + H(V)),$$

where  $\hat{\nabla}^r$  is the reference connection,  $\Delta_{G(V)}$  is the second order differential operator  $\Delta_{G(V)} = \text{Tr} \nabla G(V) \nabla$ , and  $H$  is any one-form on  $\mathcal{T}$ , with values in  $C^\infty(M)$ , satisfying  $\bar{\partial}_M H(V) = \frac{i}{2} \text{Tr} \tilde{\nabla}(G(V)\rho)$ . Such a one-form  $H$  exists and is unique up to addition of the pullback of an ordinary one-form on  $\mathcal{T}$ .

In fact, we can consider rigidity as a condition on the vector fields  $V$ , rather than considering it as a condition on the family of Kähler structures. We will then get a partial connection, which is defined in these rigid directions. This shows that as soon as the family  $\mathcal{T}$  contains points  $\sigma$  such that  $H^0(M_\sigma, S^2(T)) \neq 0$ , then the constructions provide a partial connection defined at least on some non-zero tangential directions in  $\mathcal{T}$ .

Assume now that  $M$  is compact with  $H^1(M, \mathbb{R}) = 0$ . Notice, that by the Hodge decomposition theorem,  $H^{0,1}(M) = 0$  for any Kähler structure on  $M$ .

Now we further assume, that  $\mathcal{T}$  is a complex manifold, and that  $J$  is a *holomorphic* family in the sense of Definition 14, or equivalently that  $J$  gives rise to a complex structure on  $\mathcal{T} \times M$ .

We then consider the non-corrected setting of geometric quantization of  $(M, \omega)$ , namely

$$\tilde{H}_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k) = \{s \in C^\infty(M, \mathcal{L}^k) \mid (\nabla^\mathcal{L})_\sigma^{0,1} s = 0\}.$$

Under the additional assumption, that the real first Chern class of  $(M, \omega)$  is given by

$$c_1(M, \omega) = n[\frac{\omega}{2\pi}], \quad n \in \mathbb{Z}, \quad (1)$$

there is a construction, due to Andersen ([And1]), of a Hitchin connection in the trivial bundle  $\mathcal{T} \times C^\infty(M, \mathcal{L}^k)$  over  $\mathcal{T}$ , which preserves the subbundle  $\tilde{H}^{(k)} \rightarrow \mathcal{T}$ , extending Hitchin's connection constructed in [Hit]. Now when (1) is satisfied, we are able to give an explicit formula for the one-form  $H$ , namely

$$H(V) = -2nV'[F] - \partial_M F G(V) \partial_M F - \text{Tr} \tilde{\nabla}(G(V) \partial_M F),$$

where  $F$  is any smooth family of Ricci potentials on  $M$  over  $\mathcal{T}$ . Moreover the following theorem says, that if we choose the right normalization of the Ricci potentials, we can compare the Hitchin connection given by Theorem 1 with the one constructed by Andersen in the non-corrected case, and in fact they agree.

**Theorem 2.** *Let  $(M, \omega)$  be a compact, prequantizable, symplectic manifold with vanishing second Stiefel-Whitney class, and  $H^1(M, \mathbb{R}) = 0$ . Further let  $J$  be a rigid, holomorphic family of Kähler structures on  $M$  parametrized by a*

complex manifold  $\mathcal{T}$ . Assume that the first Chern class of  $(M, \omega)$  is divisible by an integer  $n$  and that its image in  $H^2(M, \mathbb{R})$  satisfies

$$c_1(M, \omega) = n \left[ \frac{\omega}{2\pi} \right].$$

Then around every point  $\sigma \in \mathcal{T}$ , there exists an open neighbourhood  $U$ , a local smooth family  $\tilde{F}$  of Ricci potentials on  $M$  over  $U$  and an isomorphism of vector bundles over  $U$

$$\varphi: \tilde{H}^{(k-n/2)}|_U \rightarrow H^{(k)}|_U,$$

such that

$$\varphi^* \nabla = \tilde{\nabla},$$

where  $\varphi^* \nabla$  is the pullback of the Hitchin connection given by Theorem 1, and  $\tilde{\nabla}$  is the Hitchin connection in  $\tilde{H}^{(k-n/2)}$  constructed in [And1], both of which are expressed in terms of  $\tilde{F}$ .

We plan to address the computation of the curvature and removal of the rigidity condition in a forthcoming publication. Also, we find it interesting to analyze the relation between the connection constructed in this paper and the “ $L^2$ -induced” constructed by Charles in [Cha]. Further we intend to consider this new construction in the moduli space setting, in which Hitchin originally constructed his connection, and which was applied further by Andersen in [And5].

Further, we find it very interesting to explore the role of Toeplitz operators and their relation to the Hitchin connection constructed in the general setting considered in this paper. In particular it would be interesting to understand if the results in [And1], [And2] and [And3] can be generalized to this setting. For the first steps in this direction see also [And4].

This paper is organized as follows. In section 2 we introduce half-form corrected geometric quantization and the notion of metaplectic structure. Section 3 is devoted to the reference connection and the calculation of its curvature. In section 4 we derive an equation that the Hitchin connection should satisfy. Then we give a solution to this equation and prove Theorem 1. Finally, in section 5, we study the relation between our construction and the construction of [And1] in the non-corrected case, culminating with a proof of Theorem 2.

## 2 Half-form Quantization and Metaplectic Structure

Consider an almost complex structure  $J$  on  $M$ , which is compatible with the symplectic structure in the sense that

$$g_J(X, Y) = \omega(X, JY)$$

defines a Riemannian metric on  $M$ . We shall denote the resulting Riemannian manifold by  $M_J$ .

The almost complex structure  $J$  induces a splitting

$$TM_{\mathbb{C}} = T_J \oplus \bar{T}_J$$

of the complexified tangent bundle into the eigenspaces of  $J$  corresponding to the eigenvalues  $i$  and  $-i$  respectively. This splitting is explicitly given by the projections onto each summand

$$\begin{aligned} \pi_J^{1,0} &= \frac{1}{2}(\text{Id} - iJ) & T_J &= \text{Im}(\pi_J^{1,0}) \\ \pi_J^{0,1} &= \frac{1}{2}(\text{Id} + iJ) & \bar{T}_J &= \text{Im}(\pi_J^{0,1}). \end{aligned} \tag{2}$$

The fact that  $T_J$  and  $\bar{T}_J$  are the eigenspaces of  $J$ , corresponding to the eigenvalues  $i$  respectively  $-i$ , is easily verified from these formulas. Very often we shall use the notation  $X' = \pi_J^{1,0}X$  and  $X'' = \pi_J^{0,1}X$  for vector fields  $X$  on  $M$ .

Tensors, such as the symplectic form and associated metric, are extended complex linearly to  $TM_{\mathbb{C}}$ .

We recall that the first Chern class  $c_1(M_J)$  is equal to minus the first Chern class of the canonical line bundle

$$K_J = \bigwedge^m T_J^*.$$

By integrality,  $c_1(M_J)$  is independent of  $J$  since the space of compatible almost complex structures on  $(M, \omega)$  is contractible. Thus, the first Chern class is an invariant of the symplectic manifold rather than the almost complex one.

Let us assume, that the second Stiefel-Whitney class  $w_2(M)$  vanishes. Since the reduction modulo 2 of the first Chern class, that is the image of  $c_1(M)$  under the map  $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2)$ , is equal to the second Stiefel-Whitney class, this implies that the first Chern class of  $M$  is even. Thus the first Chern class of  $K_J$  is even, which is equivalent to the existence of a square root  $\delta_J$  of  $K_J$ . We shall see later that the choice of such a  $\delta_J$  determines

a square root of the canonical line bundle for every other almost complex structure on  $M$ .

The metric on  $M_J$  gives rise to the Levi-Civita connection  $\tilde{\nabla}_J$ . As usual we get an induced metric and compatible connection in all tensor bundles over  $M$ , and we shall denote all of these by  $g_J$  and  $\tilde{\nabla}_J$  as well.

The metric also induces a Hermitian structure  $h_J^T$  in  $T_J$  given by

$$h_J^T(X, Y) = g_J(X, \bar{Y}),$$

for any vectors  $X$  and  $Y$  in  $T_J$ . If we further assume that  $J$  is parallel, with respect to the Levi-Civita connection  $\tilde{\nabla}_J$ , then  $J$  must be integrable and  $M_J$  Kähler. In this case  $\tilde{\nabla}_J$  preserves the holomorphic tangent bundle  $T_J$  inducing a connection  $\nabla_J^T$  compatible with  $h_J^T$ . These in turn induce a Hermitian structure  $h_J^K$  and compatible connection  $\nabla_J^K$  in the canonical line bundle  $K_J$ .

The Ricci tensor  $r_J$  on  $M_J$  is given by the following trace of the Kähler curvature

$$r_J(X, Y) = \text{Tr}(Z \mapsto \tilde{R}(Z, X)Y),$$

and the Ricci form  $\rho_J$  is the associated (1,1)-form given by

$$\rho_J(X, Y) = r(JX, Y).$$

We recall for future use that the canonical line bundle  $K_J$  has curvature  $i\rho_J$ .

Finally  $h_J^K$  and  $\nabla_J^K$  induce a Hermitian structure  $h_J^\delta$  and compatible connection  $\nabla_J^\delta$  in the line bundle  $\delta_J$ .

**Definition 3.** A prequantum line bundle over the symplectic manifold  $(M, \omega)$  is a Hermitian line bundle  $\mathcal{L}$  with a compatible connection  $\nabla^\mathcal{L}$  of curvature

$$R_{\nabla^\mathcal{L}} = -i\omega,$$

where  $R_{\nabla}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ . Such a triple  $(\mathcal{L}, h^\mathcal{L}, \nabla^\mathcal{L})$  is denoted a prequantum line bundle, and we say that the symplectic manifold is prequantizable if it admits such a bundle.

Evidently, a necessary condition for the existence of a prequantum line bundle is that the class  $[\frac{\omega}{2\pi}]$  in  $H^2(M, \mathbb{R})$  is integral, and in fact this is also sufficient. Moreover, inequivalent choices of prequantum line bundles are parametrized by the first cohomology  $H^1(M, U(1))$  with coefficients in the circle group  $U(1) \subset \mathbb{C}$  (see for instance [Woo]). We shall assume that  $M$  is prequantizable, and fix a prequantum line bundle  $(\mathcal{L}, h, \nabla^\mathcal{L})$ .

Now  $h^{\mathcal{L}}$  and  $h_J^{\delta}$  induce a Hermitian structure  $h_J$  in the line bundle  $\mathcal{L}^k \otimes \delta_J$ , and we have a compatible connection  $\nabla_J$ , induced by  $\nabla^{\mathcal{L}}$  and  $\nabla_J^{\delta}$ . Since  $\mathcal{L}^k \otimes \delta_J$  has curvature  $-ik\omega + \frac{i}{2}\rho_J$ , which is of type (1,1), the operator

$$\nabla_J^{0,1} = \pi_J^{0,1} \nabla_J$$

defines a  $\bar{\partial}$ -operator in  $\mathcal{L}^k \otimes \delta_J$ , making this a holomorphic line bundle over  $M_J$  (see e.g. [AB]). If we consider the space  $\mathcal{H}_J^{(k)} = C^\infty(M, \mathcal{L}^k \otimes \delta_J)$  of smooth sections, then the operator  $\nabla_J^{0,1}$  gives rise to the subspace  $H_J^{(k)}$  of holomorphic sections

$$H_J^{(k)} = H^0(M_J, \mathcal{L}^k \otimes \delta_J) = \{s \in C^\infty(M_J, \mathcal{L}^k \otimes \delta_J) \mid \nabla_J^{0,1}s = 0\}.$$

We can define a Hermitian inner product on this space by

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M h^{\mathcal{L}}(s_1, s_2) \omega^m,$$

and if we consider the space of square integrable functions we obtain a Hilbert space. This is the Hilbert space resulting from the half-form corrected geometric quantization of the Kähler manifold  $M_J$ .

We will construct a connection in  $\mathcal{H}^{(k)}$  and prove, that under certain conditions this connection preserves the infinitesimal condition for being contained in the subspaces  $H_J^{(k)}$ . From this we conclude, that the spaces  $H_J^{(k)}$  form a vector bundle over a manifold that parametrizes choices of  $J$ , and the fibers  $H_J^{(k)}$  are related using parallel translation of the induced connection, which we will call the Hitchin connection.

To be able to do this, we should pay closer attention to the way we choose the half-form bundle  $\delta_J$ . Clearly, there is more than one choice of a square root of  $K_J$  (when it exists), and we would like to choose  $\delta_J$  in a unified way for different  $J$ . This is where the notion of a metaplectic structure comes into the picture.

Consider the positive Lagrangian Grassmannian  $L^+M$  consisting of pairs  $(p, J_p)$ , where  $p \in M$  and  $J_p$  is a compatible almost complex structure on the tangent space  $T_p M$ . This space has the structure of a smooth bundle over  $M$ , with the obvious projection, and with sections corresponding precisely to almost complex structures on  $M$ .

At each point  $(p, J_p) \in L^+M$ , we can consider the one dimensional space  $K_{J_p} = \bigwedge^m T_{J_p}^*$ . These form a smooth bundle  $K$  over  $L^+M$ , and the pullback



by a section of  $L^+M$  yields the canonical line bundle associated to the almost complex structure on  $M$  given by the section.

We want to find a square root  $\delta \rightarrow L^+M$  of the bundle  $K \rightarrow L^+M$ . Such a square root is called a metaplectic structure on  $M$ . Since  $L^+M$  has contractible fibers, we can find local trivializations of  $K$  with constant transition functions along the fibers. The construction of a metaplectic structure on  $M$  amounts to choosing square roots of these transition functions in such a way that they still satisfy the cocycle conditions. But since the transition functions are constant along the fibers, we only have to choose a square root at a single point in each fiber. In other words, a square root  $\delta_J$  of  $K_J$ , for a single almost complex structure  $J$  on  $M$ , determines a metaplectic structure. We summarize this in a proposition.

**Proposition 4.** *Let  $M$  be a manifold with vanishing second Stiefel-Whitney class, and let  $\omega$  be any symplectic structure on  $M$ . Then  $(M, \omega)$  admits a metaplectic structure  $\delta \rightarrow L^+M$ .*

For the rest of this paper, we shall assume that  $M$  satisfies the conditions of this proposition, and fix a metaplectic structure  $\delta$ . In this way, for every almost complex structure  $J$  on  $M$ , viewed as a section of  $L^+M$ , we have a canonical choice of square root of the canonical line bundle, given as the pullback of  $\delta$  by  $J$ .

### 3 The Reference Connection

Returning to the setup of the introduction, consider a manifold  $\mathcal{T}$ , and assume that we have a smooth family  $J: \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM))$  of Kähler structures on  $M$ , parametrized by  $\mathcal{T}$ . More precisely  $J$  is a smooth section of the pullback bundle  $\pi_M^* \text{End}(TM) \rightarrow \mathcal{T} \times M$ , where  $\pi_M: \mathcal{T} \times M \rightarrow M$  is the projection, such that for every  $\sigma \in \mathcal{T}$ , the endomorphism  $J_\sigma$  defines a complex structure on  $M$ , turning this into a Kähler manifold  $M_\sigma$ . As in the previous section, the Kähler metric is given by

$$g_\sigma(X, Y) = \omega(X, J_\sigma Y),$$

and  $J_\sigma$  induces a splitting  $TM_\mathbb{C} = T_\sigma \oplus \bar{T}_\sigma$ . Also we write  $X'_\sigma = \pi_\sigma^{1,0} X$  and  $X''_\sigma = \pi_\sigma^{0,1} X$  for any vector field  $X$  on  $M$ .

Viewing the family  $J$  as a map  $\mathcal{T} \times M \rightarrow L^+M$ , we get a smooth bundle  $\delta \rightarrow \mathcal{T} \times M$ , by pulling back the metaplectic structure on  $M$ . For any  $\sigma \in \mathcal{T}$ ,

the restriction

$$\delta_\sigma = \delta|_{\{\sigma\} \times M} \rightarrow M$$

is a square root of the canonical line bundle  $K_\sigma$  on  $M_\sigma$ . Moreover the Hermitian structure  $h_\sigma^\delta = h_{J_\sigma}^\delta$  in  $\delta_\sigma$  gives rise to a Hermitian structure  $h^\delta$  on  $\delta$ . Let  $\pi_M : \mathcal{T} \times M \rightarrow M$  denote the projection and define

$$\hat{\mathcal{L}} = \pi_M^* \mathcal{L} = \mathcal{T} \times \mathcal{L},$$

with Hermitian metric  $\hat{h}^\mathcal{L} = \pi_M^* h^\mathcal{L}$ . When objects are extended to the product  $\mathcal{T} \times M$ , we shall often use a hat to indicate, that we are dealing with the extended object. Then  $\hat{\mathcal{L}} \otimes \delta$  becomes a smooth line bundle over  $\mathcal{T} \times M$  with Hermitian metric  $\hat{h}$  induced by  $\hat{h}^\mathcal{L}$  and  $h^\delta$ .

As in the previous section we consider the space  $\mathcal{H}_\sigma^{(k)} = C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$ , in which the connection  $\nabla_{J_\sigma}$ , which we shall denote by  $\nabla_\sigma$ , gives rise to the subspace of holomorphic sections

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) = \{s \in \mathcal{H}_\sigma^{(k)} \mid \nabla_\sigma^{0,1} s = 0\}.$$

In fact the spaces  $\mathcal{H}_\sigma^{(k)}$  form a smooth vector bundle  $\mathcal{H}^{(k)}$  over  $\mathcal{T}$ . We will construct a connection in  $\mathcal{H}^{(k)}$  which preserves the spaces  $H_\sigma^{(k)}$ , thereby proving that these form a smooth subbundle  $H^{(k)}$  of  $\mathcal{H}^{(k)}$ , and at the same time giving a connection in  $H^{(k)}$ .

First we define a connection  $\hat{\nabla}^\mathcal{L}$  in  $\hat{\mathcal{L}}$  simply by extending  $\nabla^\mathcal{L}$  as the trivial connection in directions tangent to  $\mathcal{T}$ , i.e.  $\hat{\nabla}^\mathcal{L}$  is the pullback connection in the pullback bundle  $\hat{\mathcal{L}}$ . Concretely, if  $X$  is a vector field on  $\mathcal{T} \times M$ , which is tangent to  $M$ , and  $s$  is a section of  $\hat{\mathcal{L}}$ , then we define

$$(\hat{\nabla}_X^\mathcal{L} s)_{(\sigma,p)} = (\nabla_X^\mathcal{L} s)_p.$$

For any vector field  $V$  on  $\mathcal{T} \times M$ , which is tangent to  $\mathcal{T}$ , we have that

$$(\hat{\nabla}_V^\mathcal{L} s)_{(\sigma,p)} = V[s_p]_\sigma.$$

Here  $V[s_p]_\sigma$  denotes differentiation at  $\sigma \in \mathcal{T}$  along  $V$  of  $s_p$ , as a section of the trivial bundle  $\mathcal{T} \times \mathcal{L}_p$ .

Now  $\hat{\nabla}^\mathcal{L}$  is easily seen to be compatible with the Hermitian structure  $\hat{h}^\mathcal{L}$ , and for future reference we give the curvature, which is easily calculated.

**Lemma 5.** *The curvature of  $\hat{\nabla}^{\mathcal{L}}$  is given by*

$$R_{\hat{\nabla}^{\mathcal{L}}} = \pi_M^* R_{\nabla^{\mathcal{L}}} = -i\pi_M^* \omega,$$

where  $\pi_M: \mathcal{T} \times M \rightarrow M$  denotes the projection.

Next we define a connection  $\hat{\nabla}^T$  in the bundle  $T \rightarrow \mathcal{T} \times M$  in the following way. In the directions tangent to  $M$ , simply take  $\hat{\nabla}^T$  to be the connection  $\nabla^T$  induced from the Levi-Civita connection. More explicitly we define, for any section  $Y$  of  $T$  and any vector  $X \in T_p M$ ,

$$(\hat{\nabla}_X^T Y)_{(\sigma,p)} = ((\nabla_X^T) Y_\sigma)_p, \quad (3)$$

where  $\nabla_\sigma^T$  denotes  $\nabla_{J_\sigma}^T$ . For the directions along  $\mathcal{T}$ , we let  $V \in T_\sigma \mathcal{T}$  be any vector on  $\mathcal{T}$  and define

$$(\hat{\nabla}_V^T Y)_{(\sigma,p)} = \pi_\sigma^{1,0} V[Y_p]_\sigma, \quad (4)$$

for any section  $Y$  of  $T$ , where  $V[Y_p]$  denotes differentiation of  $Y_p$  in the trivial bundle  $\mathcal{T} \times T_p M_{\mathbb{C}}$ , and  $\pi_\sigma^{1,0}: \mathcal{T} \times T M_{\mathbb{C}} \rightarrow T_\sigma$  is the projection.

Now  $\hat{\nabla}^T$  induces a connection  $\hat{\nabla}^K$  in  $K = \bigwedge^m T^*$ , which in turn induces a connection  $\hat{\nabla}^\delta$  in the square root  $\delta$ . With the help of the connection  $\hat{\nabla}^{\mathcal{L}}$ , this induces a connection  $\hat{\nabla}^r$  in the line bundle  $\hat{\mathcal{L}}^k \otimes \delta$ .

**Definition 6.** *The connection*

$$\hat{\nabla}^r = (\hat{\nabla}^{\mathcal{L}})^{\otimes k} \otimes \text{Id} + \text{Id} \otimes \hat{\nabla}^\delta$$

in  $\hat{\mathcal{L}}^k \otimes \delta \rightarrow \mathcal{T} \times M$  is called the reference connection.

Notice how the reference connection induces a connection in  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ . Indeed, for any section  $s$  of  $\mathcal{H}^{(k)}$  (which is the same as a section of  $\hat{\mathcal{L}}^k \otimes \delta$  over  $\mathcal{T} \times M$ ) and any vector field  $V$  tangent to  $\mathcal{T}$ , it is simply given by  $\hat{\nabla}_V^r s$ . Moreover, if we restrict to a point  $\sigma \in \mathcal{T}$  and take  $X$  to be a vector field tangent to  $M$ , then  $(\hat{\nabla}_X^r s)_\sigma = (\nabla_X)_\sigma s_\sigma$ , so the reference connection is a unified description of a connection in  $\mathcal{H}^{(k)}$  and the connections in the bundles  $\mathcal{L}^k \otimes \delta_\sigma \rightarrow M$ .

## Curvature

Later we shall have need for the curvature of the reference connection, which is given by Proposition 7, 8 and 9 below.

**Proposition 7.** *For vector fields  $X$  and  $Y$ , tangent to  $M$ , we have*

$$R_{\hat{\nabla}^r}(X, Y) = -ik\omega(X, Y) + \frac{i}{2}\rho(X, Y), \quad (5)$$

where  $\rho_\sigma$  denotes the Ricci form on  $M_\sigma$ .

*Proof.* This follows immediately by the curvature of prequantum line bundles and the standard fact that the canonical line bundle  $K_\sigma$  over  $M_\sigma$  has curvature  $i\rho_\sigma$ .  $\square$

Before giving the curvature in the mixed directions, we introduce some more notation. Since the symplectic form is non-degenerate, it induces an isomorphism

$$i_\omega: TM_{\mathbb{C}} \rightarrow TM_{\mathbb{C}}^*,$$

by contraction in the first entry. Moreover  $\omega$  is  $J$ -invariant, or equivalently of type (1,1), which implies that  $i_\omega$  interchanges types. Similarly the metric induces a type-interchanging isomorphism  $i_g: TM_{\mathbb{C}} \rightarrow TM_{\mathbb{C}}^*$ , and the two are related by  $i_g = -Ji_\omega$ .

For any vector field  $V$  tangent to  $\mathcal{T}$ , we can differentiate the family of complex structures in the direction of  $V$  and obtain

$$V[J]: \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM_{\mathbb{C}})).$$

By differentiation of the identity  $J^2 = -\text{Id}$ , we see that  $V[J]$  anticommutes with  $J$ . This in turn implies that  $V[J]_\sigma$  interchanges types on  $M_\sigma$ , whence it decomposes as

$$V[J]_\sigma = V[J]'_\sigma + V[J]''_\sigma,$$

where  $V[J]'_\sigma \in C^\infty(M, \bar{T}_\sigma^* \otimes T_\sigma)$  and  $V[J]''_\sigma \in C^\infty(M, T_\sigma^* \otimes \bar{T}_\sigma)$ .

Now define  $\tilde{G}(V) \in C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$  by the relation

$$V[J] = (\text{Id} \otimes i_\omega)(\tilde{G}(V))$$

for all vector fields  $V$ . We use the notation

$$\tilde{G}(V)\omega = (\text{Id} \otimes i_\omega)(\tilde{G}(V)).$$

The way to interpret this, is to trace the right contravariant part of  $\tilde{G}(V)$  with the left covariant part of  $\omega$ , as prescribed by  $(\text{Id} \otimes i_\omega)(\tilde{G}(V))$ . Now observe, that the combined types of  $V[J]$  and  $\omega$  yield a decomposition

$$\tilde{G}(V) = G(V) + \bar{G}(V), \quad (6)$$

for all real vector fields  $V$  on  $\mathcal{T}$ , where  $G(V)_\sigma \in C^\infty(M, T_\sigma \otimes T_\sigma)$  and  $\bar{G}(V)_\sigma \in C^\infty(M, \bar{T}_\sigma \otimes \bar{T}_\sigma)$ . Differentiating the definition,  $g = \omega J$ , of the metric along  $V$  we have

$$V[g] = \omega V[J] = \omega \tilde{G}(V) \omega = -(i_\omega \otimes i_\omega)(\tilde{G}(V)). \quad (7)$$

Once again, notice how the notation  $\omega \tilde{G}(V) \omega$  is used to denote tracing the right covariant part of  $\omega$  with the left contravariant part of  $\tilde{G}(V)$ , as well as tracing the right contravariant part of  $\tilde{G}(V)$  with the left covariant part of  $\omega$ . Since  $g$  is symmetric, so is  $V[g]$ , which implies that  $G(V)_\sigma \in C^\infty(M, S^2(T_\sigma))$  and  $\bar{G}(V)_\sigma \in C^\infty(M, S^2(\bar{T}_\sigma))$ .

By a small calculation, we obtain another useful formula for the connection  $\hat{\nabla}^T$  in the directions tangent to  $\mathcal{T}$ . Indeed, we have that

$$\hat{\nabla}_V^T Y = V[\pi^{1,0} Y] - V[\pi^{1,0}] Y = V[Y] + \frac{i}{2} V[J] Y, \quad (8)$$

for any section  $Y$  of  $T$ .

Now we are ready to calculate the curvature of the reference connection in the remaining directions. To do this we recall the general fact, which was already implicitly used to find the curvature of the half-form bundle, that the curvature of  $\hat{\nabla}^\delta$  is given by

$$R_{\hat{\nabla}^\delta} = -\frac{1}{2} \text{Tr } R_{\hat{\nabla}^T}, \quad (9)$$

where we trace the endomorphism part of  $R_{\hat{\nabla}^T} \in \Omega^2(T \times M, \text{End}(T))$ . The change of sign appears when we induce  $\hat{\nabla}^T$  in  $T^*$ , the trace appears when we induce in  $K = \bigwedge^m T^*$ , and the division by two appears when we induce in  $\delta$ . Then we have

**Proposition 8.** *For vector fields  $V$  and  $W$  tangent to  $\mathcal{T}$  we have*

$$R_{\hat{\nabla}^r}(V, W) = 0 \quad (10)$$

*Proof.* Take  $V$  and  $W$  to be pullbacks of vector fields on  $\mathcal{T}$  which satisfy that  $[V, W] = 0$ . Then using (8), we find that

$$\begin{aligned} \hat{\nabla}_V^T \hat{\nabla}_W^T Y &= \hat{\nabla}_V^T (W[Y] + \frac{i}{2} W[J] Y) \\ &= VW[Y] + \frac{i}{2} VW[J] Y + \frac{i}{2} W[J] V[Y] + \frac{i}{2} V[J] W[Y] - \frac{1}{4} V[J] W[J] Y. \end{aligned}$$

Using that  $V$  and  $W$  commute we get

$$\begin{aligned} R_{\hat{\nabla}^T}(V, W)Y &= \hat{\nabla}_V^T \hat{\nabla}_W^T Y - \hat{\nabla}_W^T \hat{\nabla}_V^T Y \\ &= -\frac{1}{4}(V[J]W[J] - W[J]V[J])Y \\ &= -\frac{1}{4}[V[J], W[J]]Y, \end{aligned}$$

and so by (9) we get

$$R_{\hat{\nabla}^r}(V, W) = R_{\hat{\nabla}^L}^{(k)}(V, W) - \frac{1}{2} \text{Tr } R_{\hat{\nabla}^T}(V, W) = 0,$$

as desired, since  $R_{\hat{\nabla}^L}(V, W) = 0$  and  $R_{\hat{\nabla}^T}(V, W)$  is a commutator.  $\square$

Now we calculate the curvature of the reference connection in the mixed directions.

**Proposition 9.** *For vector fields  $V$  and  $X$ , tangent to  $\mathcal{T}$  and  $M$  respectively, we have*

$$R_{\hat{\nabla}^r}(V, X) = \frac{i}{4} \text{Tr } \tilde{\nabla}(\tilde{G}(V))\omega X \quad (11)$$

*Proof.* First we calculate the curvature of  $\hat{\nabla}^T$ . Let  $X$  and  $V$  be pullbacks of real vector fields on  $M$  and  $\mathcal{T}$  respectively, and let  $Y$  be any section of  $T$ . Then we get

$$\begin{aligned} R_{\hat{\nabla}^T}(V, X)Y &= \hat{\nabla}_V^T \hat{\nabla}_X^T Y - \hat{\nabla}_X^T \hat{\nabla}_V^T Y \\ &= \pi^{1,0}V[\tilde{\nabla}_X Y] - \tilde{\nabla}_X \pi^{1,0}V[Y] \\ &= \pi^{1,0}V[\tilde{\nabla}_X Y] - \pi^{1,0}\tilde{\nabla}_X V[Y] \\ &= \pi^{1,0}V[\tilde{\nabla}]_X Y \end{aligned}$$

By Theorem 1.174 in [Bes], we get that the variation of the Levi-Civita connection in the tangent bundle is a symmetric (2,1)-tensor given by

$$\begin{aligned} g(V[\tilde{\nabla}]_X Y, Z) &= \frac{1}{2}(\tilde{\nabla}_X(V[g])(Y, Z) \\ &\quad + \tilde{\nabla}_Y(V[g])(X, Z) \\ &\quad - \tilde{\nabla}_Z(V[g])(X, Y)) \end{aligned} \quad (12)$$

for vector fields  $X$ ,  $Y$  and  $Z$  on  $M$  and  $V$  on  $\mathcal{T}$ . We focus our attention on a point  $p \in M$ , and let  $e_1, \dots, e_m$  be a basis of  $T_p M$  satisfying the orthogonality condition that  $g(e'_j, e''_l) = \delta_{jl}$ . Then

$$\text{Tr } R_{\hat{\nabla}^T}(V, X) = \text{Tr } \pi^{1,0}V[\tilde{\nabla}]_X \pi^{1,0} = \sum_{\nu} g(V[\tilde{\nabla}]_X e'_\nu, e''_\nu).$$

But taking into account the type of  $V[g]$ , and the fact that  $\tilde{\nabla}$  preserves types, we get

$$\begin{aligned}
g(V[\tilde{\nabla}]_X e'_\nu, e''_\nu) &= \frac{1}{2} \tilde{\nabla}_{e'_\nu}(V[g])(X, e''_\nu) - \frac{1}{2} \tilde{\nabla}_{e''_\nu}(V[g])(X, e'_\nu) \\
&= \frac{1}{2} X \omega \tilde{\nabla}_{e'_\nu}(\tilde{G}(V)) \omega e''_\nu - \frac{1}{2} X \omega \tilde{\nabla}_{e''_\nu}(\tilde{G}(V)) \omega e'_\nu \\
&= \frac{i}{2} X \omega \tilde{\nabla}_{e'_\nu}(G(V)) g e''_\nu + \frac{i}{2} X \omega \tilde{\nabla}_{e''_\nu}(\bar{G}(V)) g e'_\nu \\
&= -\frac{i}{2} g(\tilde{\nabla}_{e'_\nu}(G(V)) \omega X, e''_\nu) - \frac{i}{2} g(\tilde{\nabla}_{e''_\nu}(\bar{G}(V)) \omega X, e'_\nu).
\end{aligned}$$

Summing over  $\nu$ , we conclude that

$$\begin{aligned}
\text{Tr } R_{\hat{\nabla}^T}(V, X) &= -\frac{i}{2} \text{Tr } \tilde{\nabla}(G(V)) \omega X - \frac{i}{2} \text{Tr } \tilde{\nabla}(\bar{G}(V)) \omega X \\
&= -\frac{i}{2} \text{Tr } \tilde{\nabla}(\tilde{G}(V)) \omega X,
\end{aligned}$$

at the point  $p$  which was arbitrary. Finally we get by Lemma 5 and (9) that

$$\begin{aligned}
R_{\hat{\nabla}^r}(V, X) &= R_{\hat{\nabla}^{\mathcal{L}}}^{(k)}(V, X) - \frac{1}{2} \text{Tr } R_{\hat{\nabla}^T}(V, X) \\
&= \frac{i}{4} \text{Tr } \tilde{\nabla}(\tilde{G}(V)) \omega X,
\end{aligned}$$

which was the claim.  $\square$

## 4 The Hitchin Connection

Let  $\mathcal{D}(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$  denote the space of differential operators on  $\mathcal{H}_\sigma^{(k)} = C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$ , and consider the bundle  $\mathcal{D}(M, \hat{\mathcal{L}}^k \otimes \delta)$  over  $\mathcal{T}$  having these spaces as fibers. One could think of  $\mathcal{D}(M, \hat{\mathcal{L}}^k \otimes \delta)$  as the space of differential operators on sections of  $\hat{\mathcal{L}}^k \otimes \delta$ , which are of order zero in the directions tangent to  $\mathcal{T}$ . Then, for any one-form  $u$  on  $\mathcal{T}$  with values in  $\mathcal{D}(M, \hat{\mathcal{L}}^k \otimes \delta)$ , we have a connection  $\nabla$  in the bundle  $\mathcal{H}^{(k)} = C^\infty(M, \mathcal{L}^k \otimes \delta)$  over  $\mathcal{T}$  given by

$$\nabla_V = \hat{\nabla}_V^r + u(V),$$

for any vector field  $V$  on  $\mathcal{T}$ . Now we wish to find a  $u$  such that  $\nabla$  preserves the subspaces  $H_\sigma^{(k)}$ , thereby proving that these form a subbundle and inducing a connection in this subbundle.

**Lemma 10.** *The connection  $\nabla$  preserves  $H^{(k)}$  if and only if*

$$\nabla^{0,1}u(V)s = \frac{i}{2}V[J]\nabla s + \frac{i}{4}\text{Tr } \tilde{\nabla}(G(V))\omega s, \quad (13)$$

for all vector fields  $V$  on  $\mathcal{T}$ , and all  $s \in H^{(k)}$ .

*Proof.* Let  $X$  and  $V$  be the pullbacks of a vector field on  $M$  and  $\mathcal{T}$  respectively. Then we see that

$$[V, X''] = \frac{i}{2}V[J]X. \quad (14)$$

Now, assume that  $s \in H^{(k)}_\sigma$  and consider any extension of  $s$  to a smooth section of  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ . Then we get

$$\begin{aligned} \nabla_{X''}\nabla_V s &= \hat{\nabla}_{X''}^r \hat{\nabla}_V^r s + \nabla_{X''}u(V)s \\ &= \hat{\nabla}_V^r \hat{\nabla}_{X''}^r s - R_{\hat{\nabla}^r}(V, X'')s - \hat{\nabla}_{[V, X'']}^r s + \nabla_{X''}u(V)s \\ &= -\frac{i}{2}\nabla_{V[J]X}s - \frac{i}{4}\text{Tr}(\tilde{\nabla}(G(V))\omega X)s + \nabla_{X''}u(V)s, \end{aligned}$$

at the point  $\sigma \in \mathcal{T}$ , where we used (14) and Proposition 9 for the last equality. This tells us, that  $\nabla$  preserves  $H^{(k)}$  if and only if  $u$  satisfies the equation in the lemma.  $\square$

For any vector field  $V$  tangent to  $\mathcal{T}$ , the tensor  $G(V)_\sigma \in C^\infty(M_\sigma, S^2(T_\sigma))$  induces a linear map  $G(V)_\sigma: TM_\mathbb{C}^* \rightarrow TM_\mathbb{C}$ , by the formula

$$\alpha \mapsto \text{Tr}(G(V)_\sigma \otimes \alpha) = G(V)_\sigma \alpha.$$

Obviously this is in fact a map  $G(V)_\sigma: T_\sigma^* \rightarrow T_\sigma$ . We then define a second order operator  $\Delta_{G(V)_\sigma} \in \mathcal{D}(M, \mathcal{L}^k \otimes \delta_\sigma)$  by  $\Delta_{G(V)_\sigma} = \text{Tr } \nabla_\sigma G(V)_\sigma \nabla_\sigma$ , or more explicitly by the diagram

$$\begin{array}{ccc} & & C^\infty(M_\sigma, TM_\mathbb{C}^* \otimes \mathcal{L}^k \otimes \delta_\sigma) \\ & \nearrow \nabla_\sigma & \downarrow G(V)_\sigma \otimes \text{Id} \otimes \text{Id} \\ C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) & & C^\infty(M_\sigma, T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma) \\ & \nwarrow \text{Tr} & \downarrow \tilde{\nabla}_\sigma \otimes \text{Id} + \text{Id} \otimes \nabla_\sigma \\ & & C^\infty(M_\sigma, TM_\mathbb{C}^* \otimes T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma) \end{array} \quad (15)$$

We shall make the additional assumption, that the family  $J$  is rigid in the sense that  $G(V)_\sigma$  should be a holomorphic section of  $S^2(T_\sigma)$  over  $M_\sigma$ .



**Definition 11.** *The family  $J$  of kähler structures on  $(M, \omega)$  is called rigid if*

$$\tilde{\nabla}_\sigma^{0,1}(G(V)_\sigma) = 0$$

*for all vector fields  $V$  tangent to  $\mathcal{T}$  and  $\sigma \in \mathcal{T}$ .*

From now on, we will for simplicity often suppress the subscription  $\sigma$  from the notation. Under this assumption we have the following lemma

**Lemma 12.** *At every point  $\sigma \in \mathcal{T}$ , the operator  $\Delta_{G(V)} = \text{Tr } \nabla G(V) \nabla$  satisfies*

$$\nabla^{0,1} \Delta_{G(V)} s = -2ik\omega G(V) \nabla s + ik \text{Tr } \tilde{\nabla}(G(V)) \omega s - \frac{i}{2} \text{Tr } \tilde{\nabla}(G(V) \rho) s$$

*for all vector fields  $V$  tangent to  $\mathcal{T}$  and all (local) holomorphic sections  $s$  of the line bundle  $\mathcal{L}^k \otimes \delta \rightarrow M$ .*

*Proof.* The proof is by direct calculation. Letting  $G$  denote  $G(V)$  we have

$$\nabla^{0,1} \Delta_G s = \nabla^{0,1} \text{Tr } \nabla G \nabla s = \text{Tr } \nabla^{0,1} \nabla G \nabla s.$$

Working further on the right side we commute the two connections, giving as ekstra terms the curvature of  $M_\sigma$  and of the line bundle  $\mathcal{L}^k \otimes \delta_\sigma$ ,

$$\nabla^{0,1} \Delta_G s = \text{Tr } \nabla \nabla^{0,1} G \nabla s - ik\omega G \nabla^{1,0} s + \frac{i}{2} \rho G \nabla s - i\rho G \nabla s.$$

Collecting the last two terms, and using the fact that  $J$  is rigid on the first, we obtain

$$\nabla^{0,1} \Delta_G s = \text{Tr } \nabla G \nabla^{0,1} \nabla s - ik\omega G \nabla s - \frac{i}{2} \rho G \nabla s.$$

Commuting the two connections, and using that  $s$  is holomorphic, we get

$$\nabla^{0,1} \Delta_G s = ik \text{Tr } \nabla G \omega s - \frac{i}{2} \text{Tr } \nabla G \rho s - ik\omega G \nabla s - \frac{i}{2} \rho G \nabla s.$$

Expanding the covariant derivatives in the first two terms by the Leibniz rule, and using the fact that  $\omega$  is parallel, we get the following, after collecting and cancelling terms

$$\nabla^{0,1} \Delta_G s = ik \text{Tr } \tilde{\nabla}(G) \omega s - 2ik\omega G \nabla s - \frac{i}{2} \text{Tr } \tilde{\nabla}(G \rho) s$$

This was the desired expression. Moreover we notice, that the above is a local computation, so that the identity is valid for local holomorphic sections of  $\mathcal{L}^k \otimes \delta$  as well.  $\square$

**Corollary 13.** *Provided that  $H^{0,1}(M) = 0$ , we have that  $\text{Tr } \tilde{\nabla}(G(V)\rho)$  is exact with respect to the  $\bar{\partial}$ -operator on  $M$ .*

*Proof.* By appealing to Lemma 12, in the case where  $k = 0$ , we get for any local holomorphic section  $s$  of  $\mathcal{L}^k \otimes \delta_\sigma \rightarrow M_\sigma$  that

$$0 = \frac{i}{2} \nabla_\sigma^{0,1} \text{Tr } \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma) s = \frac{i}{2} \bar{\partial}_\sigma (\text{Tr } \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma)) s.$$

This immediately implies that

$$0 = \bar{\partial}_\sigma (\text{Tr } \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma)),$$

and since  $H^{0,1}(M) = 0$ , the corollary follows.  $\square$

We remark, that the assumption  $H^{0,1}(M) = 0$  is satisfied for any compact Kähler manifold with  $H^1(M, \mathbb{R}) = 0$ , by the Hodge decomposition theorem.

By Corollary 13, we choose any smooth one-form  $H \in \Omega^1(\mathcal{T}, C^\infty(M))$ , such that

$$\bar{\partial}H(V) = \frac{i}{2} \text{Tr } \tilde{\nabla}(G(V)\rho), \quad (16)$$

for any vector field  $V$  on  $\mathcal{T}$ . Then finally we define

$$u(V) = \frac{1}{4k} (\Delta_{G(V)} + H(V)), \quad (17)$$

which clearly solves equation (13). Thus we have proved Theorem 1.

## 5 Relation to Non-Corrected Quantization

We now impose the same assumptions as in [And1] in order to give an explicit solution and to compare the constructed Hitchin connection with that previously constructed in [And1].

Thus, from now on  $M$  is assumed to be compact with  $H^1(M, \mathbb{R}) = 0$ . The real first Chern class of  $(M, \omega)$ , that is the image of the first Chern class in  $H^2(M, \mathbb{R})$ , is assumed to satisfy

$$c_1(M, \omega) = n \left[ \frac{\omega}{2\pi} \right], \quad (18)$$

where  $n \in \mathbb{Z}$  is some integer, which must be even by our assumption on the second Stiefel-Whitney class of  $M$ . Finally  $\mathcal{T}$  is assumed to be a complex manifold and the map  $J$  to be *holomorphic* in the following sense.

**Definition 14.** *The family  $J$ , of Kähler structures on  $M$  parametrized by  $\mathcal{T}$ , is called holomorphic if it satisfies*

$$V'[J] = V[J]' \quad \text{and} \quad V''[J] = V[J]'',$$

*for every vector field  $V$  tangent to  $\mathcal{T}$ .*

These assumptions have a number of consequences which we shall explore in the following. First, we give an alternative characterization of holomorphic families of Kähler structures.

Let  $I$  denote the integrable almost complex structure on  $\mathcal{T}$  induced by its complex structure. Then we have an almost complex structure  $\hat{J}$  on  $\mathcal{T} \times M$  defined by

$$\hat{J}(V \oplus X) = IV \oplus J_\sigma X, \quad V \oplus X \in T_{(\sigma,p)}(\mathcal{T} \times M). \quad (19)$$

The following gives another characterization of holomorphic families.

**Proposition 15.** *The family  $J$  is holomorphic if and only if  $\hat{J}$  is integrable.*

*Proof.* We show that  $J$  is holomorphic if and only if the Nijenhuis tensor for  $\hat{J}$  vanishes. By the Newlander-Nirenberg theorem this will imply the proposition (See e.g. [KN]).

Clearly the Nijenhuis tensor vanishes, when evaluated only on vectors tangent to  $\mathcal{T}$ , since  $I$  is integrable. Likewise it will vanish when evaluated only on vectors tangent to  $M$ , since  $J$  is a family of integrable almost complex structures. Thus we are left with the case of mixed directions.

Let  $X$  and  $V$  be pullbacks to  $\mathcal{T} \times M$  of vector fields on  $M$  and  $\mathcal{T}$  respectively. Then since  $X$  is constant along  $\mathcal{T}$  and  $V$  is constant along  $M$  we find that

$$[V, JX] = V[J]X. \quad (20)$$

Now consider the following evaluation of the Nijenhuis tensor

$$\begin{aligned} N(V', X) &= [IV', JX] - [V', X] - \hat{J}[IV', X] - \hat{J}[V', JX] \\ &= i[V', JX] - \hat{J}[V', JX] \\ &= iV'[J]X - JV'[J]X \\ &= 2i\pi^{0,1}V'[J]X. \end{aligned}$$

Similarly one shows, that  $N(V'', X) = -2i\pi^{1,0}V''[J]X$ . Thus we see that  $N(V, X)$  vanishes if and only if

$$\pi^{0,1}V'[J]X = 0 \quad \text{and} \quad \pi^{1,0}V''[J]X = 0.$$

This proves the proposition.  $\square$

We shall denote by  $\hat{d}$  the differential on  $\mathcal{T} \times M$ , which splits as

$$\hat{d} = d_{\mathcal{T}} + d_M$$

into the sum of the differentials on  $\mathcal{T}$  and  $M$  respectively. Similar notation is used for  $\partial$  and  $\bar{\partial}$ .

### Explicit Formula for $H(V)$

As a first consequence of our additional assumptions we are able to give an explicit formula for the one-form  $H$  in (17).

Since the curvature of the canonical line bundle  $K_{\sigma}$  is  $i\rho_{\sigma}$ , the real first Chern class of  $M_{\sigma}$  is represented by  $\frac{\rho_{\sigma}}{2\pi}$ . Since the Kähler form is harmonic, the assumption (18) is then equivalent to  $\rho_{\sigma}^H = n\omega$ , where  $\rho_{\sigma}^H$  denotes the harmonic part of the Ricci form.

Since any real exact (1,1)-form on a Kähler manifold is  $\partial\bar{\partial}$ -exact, there exists, for any  $\sigma \in \mathcal{T}$ , a real function  $F_{\sigma}$ , called a *Ricci potential*, satisfying

$$\rho_{\sigma} = \rho_{\sigma}^H + 2i\partial_{\sigma}\bar{\partial}_{\sigma}F_{\sigma}.$$

By compactness of  $M$ , any two Ricci potentials on  $M_{\sigma}$  differ by a constant. Thus choosing a particular normalization, such as

$$\int_M F_{\sigma}\omega^m = 0, \tag{21}$$

would yield a real smooth function  $F \in C^{\infty}(\mathcal{T} \times M)$ , with  $F_{\sigma}$  a Ricci potential on  $M_{\sigma}$  for every  $\sigma \in \mathcal{T}$ . Such a function shall be called a *smooth family of Ricci potentials over  $\mathcal{T}$* , and it satisfies the identity

$$\rho = n\omega + 2i\partial_M\bar{\partial}_MF. \tag{22}$$

By a *pluriharmonic family of Ricci potentials over  $\mathcal{T}$*  we mean a smooth family of Ricci potentials satisfying

$$\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F = 0. \tag{23}$$

We shall have use for this notion later, but for the moment we just consider  $F$  to be any smooth family of Ricci potentials.

We will need the following lemma, the proof of which is given in [And1].

**Lemma 16.** *Any smooth family  $F$  of Ricci potentials satisfies*

$$\bar{\partial}_MV'[F] = -\frac{i}{4}\text{Tr}\tilde{\nabla}(G(V))\omega - \frac{i}{2}\partial_MFG(V)\omega, \tag{24}$$

for any vector field  $V$  tangent to  $\mathcal{T}$ .

Then we have the following

**Lemma 17.** *Let  $F$  be a smooth family of Ricci potentials. Then the one-form  $H \in \Omega^1(\mathcal{T}, C^\infty(M))$  given by*

$$H(V) = -2nV'[F] - \partial_M F G(V) \partial_M F - \text{Tr } \tilde{\nabla}(G(V) \partial_M F)$$

*satisfies  $\bar{\partial}_M H(V) = \frac{i}{2} \text{Tr } \tilde{\nabla}(G(V) \rho)$ .*

*Proof.* Throughout this proof we shall denote  $\partial_M$  and  $\bar{\partial}_M$  for short by  $\partial$  and  $\bar{\partial}$  respectively. Since  $\omega$  is parallel, with respect to the Levi-Civita connection  $\tilde{\nabla}$ , we get

$$\begin{aligned} \text{Tr } \tilde{\nabla}(G(V) \rho) &= \text{Tr } \tilde{\nabla}(G(V)(n\omega + 2i\partial\bar{\partial}F)) \\ &= n \text{Tr } \tilde{\nabla}(G(V))\omega + 2i \text{Tr } \tilde{\nabla}(G(V)\partial\bar{\partial}F). \end{aligned}$$

Moreover, it is easily verified that

$$\begin{aligned} \text{Tr } \tilde{\nabla}(G(V)\partial\bar{\partial}F) &= -i\partial FG(V)\rho + \bar{\partial} \text{Tr } \tilde{\nabla}(G(V)\partial F) \\ &= -in\partial FG(V)\omega + 2\partial FG(V)\partial\bar{\partial}F + \bar{\partial} \text{Tr } \tilde{\nabla}(G(V)\partial F). \end{aligned}$$

Then the lemma follows by Lemma 16 and the identity

$$\bar{\partial}(\partial FG(V)\partial F) = 2\partial FG(V)\partial\bar{\partial}F,$$

which is easily verified, using the symmetry of  $G(V)$ .  $\square$

Thus under the assumptions of this section, we have a completely explicit formula for the Hitchin connection.

## Curvature of the Reference Connection Revisited

Notice that the type of  $\omega$ , and the fact that  $J$  is holomorphic, implies

$$V'[J] = V[J]' = G(V)\omega,$$

which in turn gives  $G(V) = G(V')$ . Then, having calculated the curvature of the reference connection in all directions, we see that it is of type (1,1) over  $\mathcal{T} \times M$  and thus the (0,2)-part of the curvature vanishes. This means that the reference connection defines a holomorphic structure on the line bundle  $\hat{\mathcal{L}}^k \otimes \delta$ , over the complex manifold  $\mathcal{T} \times M$ . Moreover, we observe that  $(\hat{\nabla}^r)^{0,1}$  preserves the bundle  $H^{(k)} \rightarrow \mathcal{T}$ , since  $u(V'') = 0$  solves (13). Thus the

reference connection defines a holomorphic structure on the bundle  $H^{(k)} \rightarrow \mathcal{T}$ .

We now prove that, at least locally over  $\mathcal{T}$ , the curvature of the reference connection can be expressed in terms of a pluriharmonic family of Ricci potentials.

First we have the following lemma, which is an immediate consequence of Lemma 16, by direct verification.

**Proposition 18.** *For any smooth family  $F$  of Ricci potentials and any vector fields  $V$  on  $\mathcal{T}$  and  $X$  on  $M$ , the curvature of the reference connection is given by*

$$R_{\hat{\nabla}^r}(V, X) = -\hat{\partial}\bar{\partial}F(V, X)$$

*Proof.* Let  $V$  and  $X$  be pullbacks of real vector fields on  $\mathcal{T}$  and  $M$  respectively. Then we have

$$\begin{aligned} \hat{\partial}\bar{\partial}F(X'', V') &= \hat{d}\hat{\partial}F(X'', V') \\ &= X''(\hat{\partial}F(V')) - V'(\hat{\partial}F(X'')) - \hat{\partial}F([X'', V']) \\ &= X''V'[F] + \frac{i}{2}\hat{\partial}F(V'[J]X) \\ &= X''V'[F] + \frac{i}{2}\partial_M FG(V)\omega X'' \\ &= -\frac{i}{4}\text{Tr } \tilde{\nabla}(G(V))\omega X'' \\ &= -R_{\hat{\nabla}^r}(V', X''), \end{aligned}$$

where we use Lemma 16 and Proposition 9 for the last two equalities. The case of  $X'$  and  $V''$  is similar by conjugation of the identity in Lemma 16.  $\square$

Then we have

**Theorem 19.** *Let  $(M, \omega)$  be a compact, prequantizable, symplectic manifold with the real first Chern class satisfying  $c_1(M, \omega) = n[\frac{\omega}{2\pi}]$ ,  $H^1(M, \mathbb{R}) = 0$  and vanishing second Stiefel-Whitney class. Let  $J$  be a rigid, holomorphic family of Kähler structures on  $M$ , parametrized by a complex manifold  $\mathcal{T}$ . Then for every pluriharmonic family of Ricci potentials  $\tilde{F}$  we have*

$$R_{\hat{\nabla}^r}^{(k)} = R_{\hat{\nabla}^{\mathcal{L}}}^{(k-n/2)} - \hat{\partial}\bar{\partial}\tilde{F}, \quad (25)$$

where  $R_{\hat{\nabla}^r}^{(k)}$  denotes curvature of the reference connection in  $\hat{\mathcal{L}}^k \otimes \delta$  and  $R_{\hat{\nabla}^{\mathcal{L}}}^{(k-n/2)}$  denotes the curvature of  $\hat{\nabla}^{\mathcal{L}}$  in  $\hat{\mathcal{L}}^{k-n/2}$ .

*Proof.* Let  $X$  and  $Y$  be vector fields tangent to  $M$ , and let  $V$  and  $W$  be vectorfields tangent to  $\mathcal{T}$ . Then by Proposition 7 and (22) we have that

$$\begin{aligned} R_{\hat{\nabla}^r}(X, Y) &= -ik\omega(X, Y) + \frac{i}{2}\rho(X, Y) \\ &= -i(k - \frac{n}{2})\omega(X, Y) - \partial_M \bar{\partial}_M \tilde{F}(X, Y) \\ &= R_{\hat{\nabla}^c}^{(k-n/2)}(X, Y) - \hat{\partial} \bar{\partial} \tilde{F}(X, Y). \end{aligned}$$

By Lemma 5, the curvature  $R_{\hat{\nabla}^c}^{(k-n/2)}$  vanishes in the remaining directions, and so the theorem follows from Proposition 18 and the fact that  $\tilde{F}$  is a pluriharmonic family.  $\square$

There is no guarantee that pluriharmonic families of Ricci potentials exist globally over  $\mathcal{T}$ . However we are able to prove that at least locally they do exist. First we prove

**Lemma 20.** *For any smooth family  $F$  of Ricci potentials, and any vector fields  $V$  and  $W$  on  $\mathcal{T}$ , we have*

$$0 = d_M [\hat{\partial} \bar{\partial} F(V, W)].$$

*Proof.* The first equality is just Proposition 8 so we shall prove the second. Take  $V, W$  and  $X$  to be commuting vector fields so that  $V$  and  $W$  are tangent to  $\mathcal{T}$  and  $X$  is tangent to  $M$ . Then we must prove

$$0 = X [\hat{\partial} \bar{\partial} F(V, W)].$$

Now by the differential Bianchi identity and Proposition 8 we have

$$\begin{aligned} 0 &= d^{\nabla^r} R_{\nabla^r}(X, W, V) \\ &= \nabla_X^r R_{\nabla^r}(W, V) - \nabla_W^r R_{\nabla^r}(X, V) + \nabla_V^r R_{\nabla^r}(X, W) \\ &= \nabla_V^r R_{\nabla^r}(X, W) - \nabla_W^r R_{\nabla^r}(X, V). \end{aligned}$$

Then Proposition 18 yields

$$\begin{aligned} 0 &= V[\hat{\partial} \bar{\partial} F(X, W)] - W[\hat{\partial} \bar{\partial} F(X, V)] \\ &= VXW''[F] - VWX''[F] - W XV''[F] + WVX''[F] \\ &= XVW''[F] - XWV''[F] \\ &= X[\hat{\partial} \bar{\partial} F(V, W)] \end{aligned}$$

as desired.  $\square$

This allows us to prove

**Proposition 21.** *Around every point  $\sigma \in \mathcal{T}$  there is an neighbourhood  $U$  and a pluriharmonic family  $\tilde{F}$  of Ricci potentials over  $U$ .*

*Proof.* Let  $\sigma \in \mathcal{T}$  and fix a smooth family  $F$  of Ricci potentials, say the one satisfying (21). Let  $V$  and  $W$  be vectorfields tangent to  $\mathcal{T}$ . Then by Lemma 20 it follows, that there exists a 2-form  $\alpha \in \Omega^2(\mathcal{T})$  on  $\mathcal{T}$  such that

$$0 = \hat{\partial}\bar{\partial}F(V, W) + \pi_{\mathcal{T}}^*\alpha(V, W), \quad (26)$$

where  $\pi_{\mathcal{T}}: \mathcal{T} \times M \rightarrow \mathcal{T}$  is the projection. Now from (26) it is easily shown that  $\alpha \in \Omega^2(\mathcal{T})$  is closed. Since  $\alpha$  is also of type (1,1), there exists a neighbourhood  $U$  around  $\sigma$  and a real function  $A \in C^\infty(U, \mathbb{R})$  such that  $\alpha|_U = \partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}A$ . But then we can define another smooth family of Ricci potentials over  $U$  by

$$\tilde{F} = F|_U + A,$$

and we see that

$$0 = \hat{\partial}\bar{\partial}\tilde{F}(V, W),$$

as desired. □

Thus, locally over  $\mathcal{T}$ , we can express the curvature of the reference connection by (25). Using this result, we are able to relate our construction of the Hitchin connection to a construction of Andersen ([And1]) in the non-corrected setting.

### Hitchin's Connection in Non-Corrected Quantization

We wish to relate the quantum spaces of half-form corrected quantization to the spaces of non-corrected geometric quantization, with the intent to describe, in the non-corrected setting, our construction of a Hitchin connection and relate it to the construction in [And1].

It turns out, that the choice of prequantum line bundle plays a role in this. This is because of the choice of metaplectic structure we made. We note that what we really chose was a half of  $c_1(M, \omega)$ , so all we know is that  $\delta$  is a line bundle satisfying  $2c_1(\delta) = -c_1(M, \omega)$ . So if we impose on  $(M, \omega)$  that  $n$  divides  $c_1(M, \omega)$ , we get that  $\frac{n}{2}$  divides  $c_1(\delta)$ . We will need that the prequantum line bundle is related to the metaplectic structure in a certain way, and the following lemma ensures that this is possible.



**Lemma 22.** *If  $c_1(M, \omega)$  is divisible by  $n$  in  $H^2(M, \mathbb{Z})$ , there exists a prequantum line bundle  $\mathcal{L}$  over  $M$  such that*

$$\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta).$$

*Proof.* Let  $\mathcal{L}_0$  be any prequantum line bundle on  $M$  and pick an auxiliary Kähler structure  $J$  on  $M$ . Let  $F_J$  be a Ricci potential on  $M$  and consider the line bundles  $(\mathcal{L}_0^{-n/2}, e^{F_J} h^{\mathcal{L}_0})$  and  $(\delta_J, h_J^\delta)$  over  $M$ . Then it is easily calculated, that the line bundles have the same curvature. Thus, the tensor product of the former with the dual of the latter yields a flat Hermitian line bundle  $L_1$ . Since  $c_1(\delta)$  is divisible by  $\frac{n}{2}$ , there exists a flat Hermitian line bundle  $L_2$  such that  $L_2^{n/2} \cong L_1$ . Finally the line bundle  $\mathcal{L} = \mathcal{L}_0 \otimes L_2$  has the structure of a prequantum line bundle, and  $\frac{n}{2}c_1(\mathcal{L}) = c_1(\mathcal{L}^{n/2}) = -c_1(\delta)$ . Thus  $\mathcal{L}$  is the desired prequantum line bundle.  $\square$

From now on, we will assume that our prequantum line bundle satisfies  $\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta)$ . We note, that only when  $H^2(M, \mathbb{Z})$  has torsion, is the assumption a further restriction on  $(M, \omega)$ , as otherwise the curvature determines the line bundle completely.

Next, let  $\tilde{F}$  be a local pluriharmonic family of Ricci potentials over  $U$ , with  $H^1(U) = 0$ , such that (25) is satisfied. We wish to construct an isomorphism  $\hat{\varphi}$  of holomorphic Hermitian line bundles over  $U \times M$

$$\hat{\varphi}: (\hat{\mathcal{L}}^{k-n/2}, e^{\tilde{F}} \hat{h}^{\mathcal{L}}) \rightarrow (\hat{\mathcal{L}}^k \otimes \delta, \hat{h}). \quad (27)$$

Since  $\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta)$ , the line bundles are isomorphic as complex line bundles. The obstruction to finding the structure preserving isomorphism  $\hat{\varphi}$  lies in the first cohomology of  $U \times M$ . But this is trivial by the Künneth formula, since  $H^1(U) = 0$  and  $H^1(M) = 0$  by assumption.

Moreover, it is easily seen that the pullback under  $\hat{\varphi}$  of the reference connection is given by

$$\hat{\varphi}^* \hat{\nabla}^r = \hat{\nabla}^{\mathcal{L}} + \hat{\partial} \tilde{F}, \quad (28)$$

since the right hand side is the unique Hermitian connection compatible with the holomorphic structure of  $\hat{\mathcal{L}}^{k-n/2}$ .

In the paper [And1], Andersen constructs a Hitchin connection in the bundle  $\mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ , preserving the subbundle of holomorphic sections. His construction is valid for any rigid, holomorphic family of Kähler structures on  $M$  parametrized by  $\mathcal{T}$ , provided that  $H^1(M, \mathbb{R}) = 0$  and  $c_1(M, \omega) = n[\frac{\omega}{2\pi}]$ .

Now, the existence of the isomorphism (27) enables us to compare his construction to the one presented in this paper. Thus we shall briefly recall that the Hitchin connection constructed in [And1] is given by

$$\tilde{\nabla}_V = \hat{\nabla}_V^{\mathcal{L}} + \tilde{u}(V), \quad (29)$$

where

$$\tilde{u}(V) = \frac{1}{4k + 2n} (\Delta_{G(V)}^{\mathcal{L}} + 2\nabla_{G(V)\partial\tilde{F}}^{\mathcal{L}} + 4kV'[\tilde{F}]), \quad (30)$$

and  $\Delta_{G(V)}^{\mathcal{L}}$  is the operator given by the diagram

$$\begin{array}{ccc} & & C^\infty(M_\sigma, TM_{\mathbb{C}}^* \otimes \mathcal{L}^k) \\ & \nearrow \nabla^{\mathcal{L}} & \downarrow G(V)_\sigma \otimes \text{Id} \\ C^\infty(M_\sigma, \mathcal{L}^k) & & C^\infty(M_\sigma, T_\sigma \otimes \mathcal{L}^k) \\ & \nwarrow \text{Tr} & \downarrow \hat{\nabla}_\sigma \otimes \text{Id} + \text{Id} \otimes \nabla^{\mathcal{L}} \\ & & C^\infty(M_\sigma, TM_{\mathbb{C}}^* \otimes T_\sigma \otimes \mathcal{L}^k) \end{array} \quad (31)$$

We leave it to the reader to verify, using (28), that the pullback by  $\hat{\varphi}$  of the operator  $\Delta_{G(V)}$ , acting on sections of  $\hat{\mathcal{L}}^k \otimes \delta$ , is given by

$$\hat{\varphi}^* \Delta_{G(V)} = \Delta_{G(V)}^{\mathcal{L}} + 2\nabla_{G(V)\partial_M \tilde{F}}^{\mathcal{L}} - H(V) - 2nV'[\tilde{F}], \quad (32)$$

where  $H(V)$  is given by the expression in Lemma 17, but in terms of  $\tilde{F}$ .

Furthermore, in the bundle  $\hat{\mathcal{L}}^{k-n/2}$ , the formula (30) becomes

$$\begin{aligned} \tilde{u}(V) &= \frac{1}{4k} (\Delta_{G(V)}^{\mathcal{L}} + 2\nabla_{G(V)\partial_M \tilde{F}}^{\mathcal{L}} - 2nV'[\tilde{F}]) + V'[\tilde{F}] \\ &= \frac{1}{4k} (\hat{\varphi}^* \Delta_{G(V)} + H(V)) + V'[\tilde{F}] \\ &= \hat{\varphi}^* u(V) + V'[\tilde{F}]. \end{aligned} \quad (33)$$

But this means, that the pullback of our Hitchin connection by  $\hat{\varphi}$  is given by

$$\begin{aligned} \hat{\varphi}^* \nabla_V &= \hat{\varphi}^* \hat{\nabla}_V^r + \hat{\varphi}^* u(V) \\ &= \hat{\nabla}_V^{\mathcal{L}} + V'[\tilde{F}] + \hat{\varphi}^* u(V) \\ &= \hat{\nabla}_V^{\mathcal{L}} + \tilde{u}(V) \\ &= \tilde{\nabla}_V \end{aligned} \quad (34)$$

Thus the two connections agree, and we have proved Theorem 2.

## References

- [AB] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.
- [ADPW] S. Axelrod, S. Della Pietra, and E. Witten. Geometric quantization of Chern-Simons gauge theory. *J. Differential Geom.*, 33(3):787–902, 1991.
- [And1] J. E. Andersen. Hitchin’s connection, Toeplitz operators and symmetry invariant deformation quantization. arXiv:math.DG/0611126.
- [And2] J. E. Andersen. Mapping class groups do not have Kazhdan’s property (T). arXiv:math.QA/0706.2184.
- [And3] J. E. Andersen. The Nielsen-Thurston classification of mapping classes is determined by TQFT. arXiv:math.QA/0605036.
- [And4] J. E. Andersen. Toeplitz operators and Hitchin’s connection. In *The many facets of geometry: A tribute to Nigel Hitchin*. Oxford University Press. To appear (2008).
- [And5] J. E. Andersen. Asymptotic faithfulness of the quantum  $SU(n)$  representations of the mapping class groups. *Ann. of Math. (2)*, 163(1):347–368, 2006.
- [AU1] J. E. Andersen and K. Ueno. Construction of the Reshetikhin-Turaev TQFT from conformal field theory. In preparation.
- [AU2] J. E. Andersen and K. Ueno. Modular functors are determined by their genus zero data. arXiv:math.QA/0611087.
- [AU3] J. E. Andersen and K. Ueno. Abelian conformal field theory and determinant bundles. *Internat. J. Math.*, 18(8):919–993, 2007.
- [AU4] J. E. Andersen and K. Ueno. Geometric construction of modular functors from conformal field theory. *J. Knot Theory Ramifications*, 16(2):127–202, 2007.
- [Bes] A. L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

- [BK] B. Bakalov and A. Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.
- [Cha] L. Charles. Semi-classical properties of geometric quantization with metaplectic correction. *Comm. Math. Phys.*, 270(2):445–480, 2007.
- [Fin] M. Finkelberg. An equivalence of fusion categories. *Geom. Funct. Anal.*, 6(2):249–267, 1996.
- [FMMN] C. Florentino, P. Matias, J. Mourão, and J. P. Nunes. Geometric quantization, complex structures and the coherent state transform. *J. Funct. Anal.*, 221(2):303–322, 2005.
- [Hit] N. J. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.*, 131(2):347–380, 1990.
- [KL1] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. I, II. *J. Amer. Math. Soc.*, 6(4):905–947, 949–1011, 1993.
- [KL2] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. III. *J. Amer. Math. Soc.*, 7(2):335–381, 1994.
- [KL3] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. IV. *J. Amer. Math. Soc.*, 7(2):383–453, 1994.
- [KN] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996. Reprint of the 1969 original, A Wiley-Interscience Publication.
- [Las] Y. Laszlo. Hitchin’s and WZW connections are the same. *J. Differential Geom.*, 49(3):547–576, 1998.
- [RT1] N. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.
- [RT2] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.

- [TUY] A. Tsuchiya, K. Ueno, and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, Boston, MA, 1989.
- [Wit] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.
- [Woo] N. M. J. Woodhouse. *Geometric quantization*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1992. Oxford Science Publications.